

# Salem sets, equidistribution and arithmetic progressions

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## Abstract

Arithmetic progressions of length 3 may be found in compact subsets of the reals that satisfy certain Fourier-dimensional as well as Hausdorff-dimensional requirements. It has been shown that a very similar result holds in the integers under analogous conditions, with Fourier dimension being replaced by the decay of a discrete Fourier transform. By using a construction of Salem's, we show that this correspondence can be made more precise. Specifically, we show that a subset of the integers can be mapped to a compact subset of the continuum in a way which preserves equidistribution properties as well as arithmetic progressions of arbitrary length, and vice versa. We use the method to characterise Salem sets in  $\mathbb{R}$  through discrete, equidistributed approximations. Finally, we discuss how this method sheds light on the generation of Salem sets by stationary stochastic processes.

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## 1 Introduction

In this paper, we define the Fourier-Stieltjes transform of a finite Borel measure  $\mu$  supported on a compact set  $A \subset \mathbb{R}$  as

$$\hat{\mu}(\xi) = \int_A e^{-2\pi i \xi x} d\mu(x). \quad (1.1)$$

Every set of multiplicity supports a positive, finite Radon measure whose Fourier-Stieltjes transform tends to zero at infinity. The *Fourier dimension* of a set of multiplicity is a measure of exactly how rapid the decay of the transform of that measure is.

**Definition 1.1.** *A compact set  $A \subseteq [0, 1]$  is said to have Fourier dimension  $\beta$  if*

$$\beta = \sup\{\alpha \in [0, 1] : \exists M_1^+(A) (|\hat{\mu}(\xi)|^2 = o(|\xi|^{-\alpha}))\}. \quad (1.2)$$

Here  $M_1^+(A)$  denotes the set of probability measures on the set  $A$ . Salem [10] first showed how to construct sets with a specific Fourier dimension, and in a way such that it coincides with the set's Hausdorff dimension. Hence, sets of equal Fourier and Hausdorff dimension are called *Salem sets*. It is possible for sets to have a positive Fourier dimension which is strictly smaller than their Hausdorff dimension (it can never be larger). Such sets are referred to as *Salem-type* sets.

Salem sets occur predominantly in probabilistic contexts, although non-trivial deterministic examples are known. The first such was constructed by Kaufman [3], although the interested reader should rather refer to the construction by Bluhm [1]. In general, each instance of a Salem set has had to be approached completely separately. It is part of the purpose of this paper to provide the framework for a more uniform approach to Salem sets, which is especially applicable in certain probabilistic cases.

In the paper [7], it was shown that the Fourier-transformational decay conditions guaranteeing the existence of a 3-term arithmetic progression in the integers shared a similarity with the Fourier-dimensional conditions guaranteeing such in the continuum. This result was an attempt to adapt a result by Łaba and Pramanik [5] to an integer context. Here we develop this correspondence further, and show that a similar class of sets can be considered as either compact subsets of the continuum or as infinite sequences in the

integers, and specifically that the existence of arithmetic progressions does not depend on which approach is used. To make the correspondence precise however, use of the discrete Fourier transform proved inadequate. Rather, one needs to formulate the correspondence in terms of degrees of equidistribution.

In section 2, we show how to construct subsets of  $[0, 1]$  from subsets of  $\mathbb{Z}$  which preserve notions of Hausdorff dimension. Section 3 is devoted to showing that the same mappings that preserve Hausdorff dimension also preserve Fourier dimension. In section 4, we characterise all Salem sets with an equidistribution condition, which also shows that we can obtain Salem-type sets in the integers from such sets in the continuum. Section 5 confirms that arithmetic progressions may be equivalently studied in Salem or Salem-type sets in either  $[0, 1]$  or  $\mathbb{Z}$ , which fact shows that the main results in [5] and [7] are closely related. As a consequence of the characterisation in section 4, we derive a theorem on equidistribution properties of randomly chosen intervals in section 6, which explains the ubiquity of Salem sets in certain probabilistic contexts, such as the sample path properties of Brownian motion.

## 2 Constructing sets in $[0, 1]$ from the integers

In this section we show how sets in the integers may be used to construct perfect sets in  $[0, 1]$ . This initial construction is very similar to that of Salem in [11].

We first define an integer analogue to Hausdorff dimension [7]. Throughout the paper,  $[0, N)$  denotes the interval  $\{0, 1, 2, \dots, N-1\}$  in  $\mathbb{N}$ .

**Definition 2.1.** *We say that a set  $A \subseteq \mathbb{N}$  has upper fractional density  $\alpha$  if*

$$\limsup_{N \rightarrow \infty} \frac{|A \cap [0, N)|}{N^\beta}$$

*is  $\infty$  for any  $\beta < \alpha$  and 0 for any  $\beta > \alpha$ . We can summarise this by saying that  $d_f^*(A) = \alpha$ .*

The lower fractional density can be similarly defined, and if the upper and lower fractional densities coincide we can of course simply refer to the *fractional density*. The definition can trivially be extended to all of  $\mathbb{Z}$ .

We now proceed with Salem's construction, adjusted to our needs. Let  $A$  be a subset of the non-negative integers, with an upper fractional density of  $\beta$ . Let  $\{\beta_i\}_{i=1}^\infty$  be a sequence in  $\mathbb{R}$  such that  $\beta_i \rightarrow \beta$  with  $\beta_i + 1 \leq \beta_{i+2} \leq \beta$  for all  $i \in \mathbb{N}$ . Associate to each  $i$  a number  $N_i \in \mathbb{N}$  such that

$$\frac{|A \cap [0, N_i)|}{N_i^\beta} = c_i > 0, \quad (2.3)$$

where we require the to be  $c_i$  be uniformly bounded from above, and away from 0 below. Divide  $[0, 1]$  into  $N_1$  equal intervals and choose an interval  $[i/N_1, i/N_1 + \zeta_1)$ , where  $0 < \zeta_1 \leq N_1^{-1}$ , if  $i \in A_1 = A \cap [0, N_1)$ . The set of intervals chosen is denoted by  $B_1$ . That is,

$$B_1 = \bigcup_{i \in A_1} \left[ \frac{i}{N_1}, \frac{i}{N_1} + \zeta_1 \right). \quad (2.4)$$

Inductively, define  $B_{k+1}$  by dividing the intervals belonging to  $B_k$  into  $N_{k+1}$  equal intervals and setting

$$\left[ \frac{j}{N_1 \dots N_k} + \frac{i}{N_1 \dots N_{k+1}}, \frac{j}{N_1 \dots N_k} + \frac{i}{N_1 \dots N_{k+1}} + \zeta_{k+1} \right) \in B_{k+1} \quad (2.5)$$

if

$$\left[ \frac{j}{N_1 \dots N_k}, \frac{j+1}{N_1 \dots N_k} \right) \in B_k \quad (2.6)$$

and  $i \in A_{k+1} = A \cap N_{k+1}$ , where  $0 < \zeta_{k+1} \leq (N_1 \cdots N_{k+1})^{-1}$ . More concisely, by setting  $M_k = N_1 \cdots N_k$ ,  $k \in \mathbb{N}$ ,

$$B_{k+1} = \bigcup_{j \in B_k} \bigcup_{i \in A_{k+1}} \left[ \frac{j}{M_k} + \frac{i}{M_{k+1}}, \frac{j}{M_k} + \frac{i}{M_{k+1}} + \zeta_{k+1} \right). \quad (2.7)$$

We define the set  $B$  as

$$B = \bigcap_{k=1}^{\infty} B_k. \quad (2.8)$$

For the convenience of having disjoint intervals throughout each stage of the construction, we require that the intervals lengths satisfy

$$\zeta_i = N_i^{-1} \left( 1 - \frac{1}{(k+1)^2} \right) = \eta_i N_i^{-1}, \quad i = 1, 2, \dots \quad (2.9)$$

The perfect set  $B$  obtained in this manner is denoted by  $B(\zeta_1, \zeta_2, \dots, \zeta_k, \dots)$ , although we usually do not find it necessary to explicitly indicate the parameters used in the construction.

To verify that this has Hausdorff dimension  $\beta$  is not hard to do, since the fractional density of the original subset of the integers dictates exactly how many intervals there are at each stage of the construction. Alternatively, one could use Proposition 3.2 in [6], which gives the Hausdorff dimension by way of Frostman's Lemma.

There is a way of representing such a set directly, as an infinite sum instead of using the Cantor-type construction. These methods are of course equivalent. Suppose that the set  $B$  was constructed as above. Let  $a^{(k)}(i)$  denote the  $i$ -th member of the set  $A_k$ . Each point  $x$  belonging to the set  $B$  can then be written as

$$x = \frac{a^{(1)}(\varepsilon_1)}{N_1} + \frac{\eta_1 a^{(2)}(\varepsilon_2)}{N_1 N_2} + \frac{\eta_1 \eta_2 a^{(3)}(\varepsilon_3)}{N_1 N_2 N_3} + \dots, \quad (2.10)$$

where  $\varepsilon_i$  ranges over all values  $1, 2, \dots, |A_i|$ . This representation is useful in calculating Fourier-Stieltjes transforms of measures on the set  $B$ .

### 3 Fourier correspondence between subsets of $\mathbb{N}$ and $[0, 1]$

We can construct measures on sets like those in the previous section in a generic way, and such that the decay of their Fourier-Stieltjes transforms can be determined. This is done by constructing an absolutely continuous measure during each stage and showing that they converge to a probability measure. This is again very similar to Salem's own construction, but with the marked difference that we use equidistribution conditions rather than assuming linear independence. In fact, certain of the sets used in the construction will decidedly not consist of linearly independent elements, since the existence of arithmetic progressions in them are assured when there is rapid decay of the discrete Fourier transform.

Suppose we are given a set  $A \subseteq \mathbb{N}$ . Letting  $\chi_{A_N}$  denote the characteristic function of  $A_N$ , considered as a cyclic group, the discrete Fourier transform is defined as follows:

$$\widehat{\chi}_{A_N}(k) = \frac{1}{N} \sum_{n=0}^{N-1} \chi_{A_N}(n) e^{-\frac{2\pi i k n}{N}}. \quad (3.11)$$

The correspondence we aim to achieve in this section owes its existence to an attempt to reconcile results of Laba and Pramanik [5] and the author [7]. The first guarantees 3-term arithmetic progressions in Salem-type subsets of  $[0, 1]$ :

**Theorem 3.1.** [5] *Assume that  $E \subset [0, 1]$  is a closed set which supports a probability measure  $\mu$  with the following properties:*

- (i)  $\mu([x, x + \epsilon]) \leq C_1 \epsilon^\alpha$  for all  $0 < \epsilon \leq 1$ ,

$$(ii) |\widehat{\mu}(k)| \leq C_2(1-\alpha)^{-B}|k|^{-\frac{\beta}{2}} \text{ for all } k \neq 0,$$

where  $0 < \alpha < 1$  and  $2/3 < \beta \leq 1$ . If  $\alpha > 1 - \epsilon_0$ , where  $\epsilon_0 > 0$  is a sufficiently small constant depending only on  $C_1, C_2, B, \beta$ , then  $E$  contains a non-trivial 3-term arithmetic progression.

The second shows that very similar conditions apply in  $\mathbb{N}$  to obtain such arithmetic progressions:

**Theorem 3.2.** [7] *Let  $A \subseteq \mathbb{N}$ . Suppose  $A$  satisfies the following conditions:*

(i)  *$A$  has upper fractional density  $\alpha$ , where  $\alpha > 1/2$ .*

(ii) *For some  $C > 0$ , the Fourier coefficients of the characteristic functions  $\chi_{A_N}$  of  $A_N = A \cap [0, N)$  satisfy*

$$|\widehat{\chi_{A_N}}(k)| \leq C(|k|N)^{-\beta/2}$$

*for large  $N$ , for some  $2/3 < \beta \leq 1$  satisfying  $\beta > 2 - 2\alpha$ .*

*Then  $A$  contains a non-trivial arithmetic progression of length 3.*

Since, as shown in the previous section, we can easily move from  $\mathbb{N}$  to  $[0, 1]$  in a way which preserves the integer analogue of Hausdorff dimension, the question becomes whether we can do the same with Fourier dimension. This proves problematic when we consider the discrete Fourier transform, because of the inherent periodic nature thereof. Instead, we turn to the phenomenon of equidistribution. Even in Salem's original construction, it is clear that the vital condition of linear independence actually implies a type of equidistribution condition.

The construction of the measures in this section is also slightly more deterministic in character, since Salem's method of determining the dimension of the set yields the required dimension only almost surely, whereas our adapted construction will certainly yield a measure with the required decay. It has to be admitted that this only hides the probabilistic aspect, since the most likely way to obtain sets with the given properties would be through probabilistic means.

We proceed with the construction of the measure. The sets  $A$ ,  $B_i$  and the numbers  $N_i$ ,  $\beta_i$ ,  $i \in \mathbb{N}$  are as in the previous section. At the  $k$ -th stage of the construction we introduce a continuous, non-decreasing function  $F_k : [0, 1] \rightarrow [0, 1]$ , where  $F_k(0) = 0$ , with the property that it increases linearly by  $(c_1 \dots c_k N_1^{\beta_1} \dots N_k^{\beta_k})^{-1}$  over each interval of length  $\eta_1 \dots \eta_k$  comprising  $B_k$ , with all the constants in question satisfying the requirements 2.3 and 2.9 of section 2. On intervals not part of the set  $B_k$ , the function remains constant, equal to its previously attained value on  $B_k$ . Hence, for each  $k$  we have a piecewise linear function increasing from 0 to 1 over the unit interval, which is a distribution function for a probability measure  $\mu_k$  supported on the set  $B_k$ . These functions converge pointwise to a function  $F$  which will be continuous and non-decreasing, with  $F(0) = 0$  and  $F(1) = 1$ , as can be easily seen by showing they form a Cauchy sequence with respect to the uniform norm. The measure corresponding to  $F$  will be denoted by  $\mu$ .

Our interest now lies in calculating asymptotic properties of the Fourier-Stieltjes transform of  $\mu$  as defined in 1.1. To do so, we find the transforms of the measures corresponding to the functions  $F_k$  by representing these as products of exponential sums.

We let  $d_k$  denote the number of intervals chosen in the  $k$ -th step of the construction of the set  $B = B(\xi_1, \xi_2, \dots)$  from the given set  $A$ . The numbers  $a^{(k)}(\varepsilon_j)$ ,  $k, j = 1, 2, 3, \dots$ , are as determined in 2.10. As in [11], we define the following:

$$Q^{(k)}(u) = \frac{1}{d_k} \sum_{j=1}^{d_k} e^{-2\pi i u \frac{a^{(k)}(\varepsilon_j)}{N_1 \dots N_k}}, \quad (3.12)$$

where  $d_k = |A_k|$  for all  $k = 1, 2, \dots$ . The Fourier-Stieltjes transform of the measure  $\mu$  can then be written as

$$\widehat{\mu}(u) = \lim_{m \rightarrow \infty} Q^{(1)}(u) \prod_{k=1}^m Q^{(k+1)}(\eta_1 \dots \eta_k u). \quad (3.13)$$

Note that  $Q^{(k)}$  corresponds closely to the discrete Fourier transform on integer values:

$$Q^{(k)}(m) = \frac{1}{|A_{N_k}|} \sum_{n=0}^{N_k-1} \chi_{A_{N_k}}(n) e^{-2\pi i n \frac{m}{N_k}} = \frac{N_k}{d_k} \widehat{\chi_{A_{N_k}}}(m). \quad (3.14)$$

It would seem then, that we can dictate the decay of  $\widehat{\mu}$  by requiring sufficient decay of  $\widehat{\chi_{A_{N_k}}}(m)$ . Although true, the formulation is more elegant when done as follows. We require that the  $Q^{(k)}$  satisfy

$$Q^{(k)}(m) = o_k(m^{-\frac{\beta_k}{2}}), \quad (3.15)$$

for  $m \in \mathbb{N}$ . As will be made clear in the next section, this is equivalent to an equidistribution condition on the intervals used in the construction of the set  $B$ . By Lemma 1 on p252 of [2], we can assume that  $Q^{(k)}(u) = o(u^{-\frac{\beta_k}{2}})$  for a continuous variable  $u$ . In order to estimate  $\widehat{\mu}$ , notice that

$$|\widehat{\mu}(u)| \leq \prod_{k=1}^p |Q^{(k+1)}(\eta_1 \cdots \eta_k u)|. \quad (3.16)$$

By choosing  $p$  large enough and  $u$  large compared to  $\eta_1 \cdots \eta_p$ , it is seen that

$$|\widehat{\mu}(u)| = o(u^{-\frac{\beta_p}{2}}). \quad (3.17)$$

Since this will hold for each large enough  $p$  and  $\beta_p \rightarrow \beta$  as  $p \rightarrow \infty$ , we can conclude that

$$|\widehat{\mu}(u)| = o(u^{-\frac{\beta}{2}}). \quad (3.18)$$

To show how this relates to Theorems 3.1 and 3.2, we simply use the expression 2.8. If the Fourier coefficients satisfy the decay in Theorem 3.2, the terms  $Q^{(k)}$  will satisfy

$$Q^{(k)}(m) = C \frac{N_k}{d_k} (m N_k)^{-\frac{\beta}{2}} = C c_k^{-1} m^{-\frac{\beta_k}{2}} N_k^{1-\frac{3\beta_k}{2}}, \quad 0 \leq m < N_k. \quad (3.19)$$

If  $\beta_k > 2/3$ , as is required in Theorem 3.2, and the  $c_k$  have a uniform lower bound, this means that  $Q^{(k)}(m)$  will satisfy the equidistribution condition 3.15. This in turn means that the decay conditions in Theorem 3.1 are satisfied. Since the Hausdorff-dimensional conditions are satisfied in Theorem 3.2 by assumption and in Theorem 3.1 by construction, we conclude that the construction preserves arithmetic progressions obtained under the conditions of Theorem 3.2. It will be shown in Section 5 that the construction actually preserves arithmetic progressions of arbitrary length.

The converse construction, that is, obtaining Salem-type sets in the integers from such sets in the continuum, will follow from the characterisation of Salem sets established in the next section. The arguments in Sections 3 and 4 seem to imply that the use of equidistribution provides a very natural setting for studying Salem-type phenomena.

## 4 Characterisation of Salem sets

The decay of the Fourier-Stieltjes transform of a measure is an indication of a certain uniformity of distribution of the support of the measure. We give an exact characterisation of how uniform such a distribution must be in terms of an approximation to the set by intervals with rational endpoints, in order to guarantee it is Salem. In fact, various degrees of uniformity will determine the existence of measures whose transforms have any desirable decay, limited only by the Hausdorff dimension of the set.

**Definition 4.1.** *Given a set  $A \subseteq [0, 1]$  and some  $N \in \mathbb{N}$ , we call a set  $A^N \subseteq \{0, 1/N, 2/N, \dots, (N-1)/N\}$  an  $N$ -approximation to  $A$  if, for each  $x \in A^N$ ,*

$$A \cap \left[ x, x + \frac{1}{N} \right) \neq \emptyset. \quad (4.20)$$

An  $N$ -approximation can equivalently be regarded as a collection of intervals in  $[0, 1]$ , and we shall use this interpretation interchangeably with the above definition.

A sequence  $\{x_n\}_{n \in \mathbb{N}}$  contained in  $[0, 1]$  is said to be *asymptotically equidistributed* with respect to a measure  $\nu \in M_1^+([0, 1])$  if for every interval  $I \subset [0, 1]$  whose endpoints are not mass-points of  $\nu$ ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} |\{1 \leq n \leq N : x_n \in I\}| = \nu(I). \quad (4.21)$$

The definition can trivially be extended to any compact subset of  $\mathbb{R}$ . The sequence is asymptotically distributed with respect to  $\nu$  if and only if the limits

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N e^{-2\pi i x_j m} = c_m \quad (4.22)$$

exist for each  $m \in \mathbb{Z}$ , and if the limits exist,  $c_m = \hat{\nu}(m)$  for each  $m$ ; see for instance [14], p142. We say that sequence is *equidistributed* if it is asymptotically equidistributed with respect to Lebesgue measure and  $c_m = 0$  in the above for all  $m \in \mathbb{Z} \setminus \{0\}$  [12].

In keeping with the formulation in and just after Definition 4.1, we shall say that a sequence of  $N_i$ -approximations  $\{A^{N_i} = \{a_1^i, a_2^i, \dots, a_{d_i}^i\} : i = 1, 2, \dots\}$  is *equidistributed of order  $\alpha$*  if

$$\frac{1}{d_i} \left( e^{-2\pi i a_1^i m} + e^{-2\pi i a_2^i m} + \dots + e^{-2\pi i a_{d_i}^i m} \right) = o(m^{-\frac{\alpha}{2}}) \quad (4.23)$$

for large  $i$  and each  $m \in \mathbb{Z} \setminus \{0\}$ , where  $d_i$  denotes the cardinality of  $A^{N_i}$  for each  $i \in \mathbb{N}$ .

We can now state the following theorem, which characterises one-dimensional Salem sets in terms of interval approximations. The condition on equidistribution will guarantee that a suitable measure exists on the set. Generalising the theorem to higher dimensions requires little extra work, since the definition of asymptotic equidistribution in higher dimensions is also an essentially one-dimensional concept.

**Theorem 4.1.** *A compact set  $A \subseteq [0, 1]$  is a Salem set of dimension  $\beta$  if and only if there exists a sequence  $\{A^{N_i}\}_{i=1}^\infty$  of  $N_i$ -approximations of  $A$ , some sequence  $\{\beta_i\}_{i=1}^\infty \subset \mathbb{R}$  converging to  $\beta$ , where  $\beta_i \leq \beta_{i+1} \leq \beta$ , and a bounded sequence  $\{c_i\}_{i=1}^\infty$  with  $\inf\{c_i : i \in \mathbb{N}\} > 0$ , such that*

- (i)  $|A_{N_i}| = c_i N_i^{\beta_i}$  for each  $i \in \mathbb{N}$ , and
- (ii) the sequence  $A_{N_i}$  is equidistributed of order  $\beta_i$  for each  $\beta_i < \beta$ .

**Proof.** Sufficiency is a consequence of the construction in section 2. The dimension is guaranteed by the calculation in section 3. To prove necessity, let  $A$  be a Salem set. Part (i) follows from the definition of Hausdorff dimension. Let  $\nu_\alpha$ ,  $\alpha < \beta$  be a probability measure with support in  $A$  such that  $|\widehat{\nu_\alpha}(\xi)|^2 = o(\xi^{-\alpha})$ . For such a measure  $\nu_\alpha$ , there exists an asymptotically distributed sequence  $\{x_j\}_{j=1}^\infty$  such that

$$\frac{1}{n} \left( e^{-2\pi i x_1 m} + e^{-2\pi i x_2 m} + \dots + e^{-2\pi i x_n m} \right) = o(m^{-\frac{\alpha}{2}}), \quad (4.24)$$

for each  $m \in \mathbb{Z} \setminus \{0\}$ . We can use this sequence to obtain a sequence  $A_{N_i}$  which is equidistributed to the right order. Let  $\{M_i\}_{i=1}^\infty$  be a sequence in  $\mathbb{N}$  with  $M_i < M_{i+1}$  for all  $i$ , and let  $\{\varepsilon_i\}_{i=1}^\infty$  be a sequence of positive reals with  $\varepsilon_i \downarrow 0$  as  $i \rightarrow \infty$ . Associate to each pair  $(M_i, \varepsilon_i)$  an integer  $m_i = m_i(M_i, \varepsilon_i)$  such that, for each interval  $I(i, j)$  of the form

$$I(i, j) = \left[ \frac{j}{M_i}, \frac{j+1}{M_i} \right), \quad i \in \mathbb{N}, \quad 0 \leq j \leq M_i - 1, \quad (4.25)$$

we have that

$$\left| \frac{1}{m_i} |\{1 \leq n \leq m_i : x_n \in I(i, j)\}| - \nu(I(i, j)) \right| < \varepsilon_i. \quad (4.26)$$

At stage  $i$ , associate to  $m_i$  an integer  $N_i = N_i(m_i)$  large enough so that intervals of the form  $[j/N_i, (j+1)/N_i)$ ,  $0 \leq j \leq N_i - 1$ , separates the elements of  $X_{m_i} = \{x_1, x_2, \dots, x_{m_i}\}$ . That is, each interval of such a form does not contain more than one element of  $\{x_1, x_2, \dots, x_{m_i}\}$ . Now associate to each  $x_i \in X_{m_i}$  the left-hand endpoint of the interval of length  $1/N_i$  that contains it. Denote the set so obtained by  $Y_{m_i} = \{y_1, y_2, \dots, y_{m_i}\}$ , with the obvious ordering inherited from  $X_{m_i}$ . Clearly, the elements of the set  $Y_{m_i}$  will satisfy

$$\left| \frac{1}{m_i} |\{1 \leq n \leq m_i : y_n \in I(i, j)\}| - \nu(I(i, j)) \right| < \varepsilon_i. \quad (4.27)$$

Since we can approximate the measure of any interval in  $[0, 1]$  arbitrarily finely with measures of intervals of the form  $I(i, j)$  with large enough  $i$  and  $j$ , it is clear that the set  $Y = \bigcup_{i=1}^{\infty} Y_i$  will satisfy

$$\lim_{N \rightarrow \infty} \frac{1}{N} |\{1 \leq n \leq N : y_n \in I\}| = \nu(I). \quad (4.28)$$

This is equivalent to stating that

$$\frac{1}{N} (e^{-2\pi i y_1 m} + e^{-2\pi i y_2 m} + \dots + e^{-2\pi i y_N m}) = o(m^{-\frac{\alpha}{2}}), \quad (4.29)$$

since we have that  $\widehat{\nu}(\xi) = o(\xi^{-\frac{\beta}{2}})$ . This implies that the sequence  $\{A^{N_i}\}$ , with the  $N_i = N_i(m_i)$  as determined above, is the required equidistributed sequence.  $\square$

It is now simple to obtain integer sets with certain density and Fourier properties from Salem subsets of  $[0, 1]$ , utilising the previous theorem.

**Definition 4.2.** We shall call a subset  $A$  of  $\mathbb{N}$  a Salem-type set if the set has upper fractional density  $\alpha \in [0, 1]$  and there is a strictly increasing sequence  $\{N_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$  such that, for each  $m \in \mathbb{Z} \setminus \{0\}$  and  $d_i = |A \cap [0, N_i)|$ ,

$$\frac{1}{d_i} \sum_{n=0}^{N_i-1} \chi_A(n) e^{-2\pi i \frac{mn}{N_i}} = o(m^{-\frac{\beta}{2}}), \quad (4.30)$$

for some  $\beta \in [0, 1]$ ,  $\beta \leq \alpha$ . A Salem set in the integers is one for which  $\alpha = \beta$ .

Given a Salem set  $A \subseteq [0, 1]$  with  $\dim A = \beta$ , we can construct a Salem set in the integers as follows. Suppose that  $\{A^{N_i} : i \in \mathbb{N}\}$  is a sequence of  $N_i$ -approximations for  $A$ , for some increasing sequence  $N_i \in \mathbb{N}$ . We assume for convenience that  $N_0 = 0$  and  $A^{N_0} = \emptyset$ . Then, for each  $i \geq 1$ , each  $A^{N_i}$  is of the form

$$A^{N_i} = \{a_1^{(N_i)} N_i^{-1}, a_2^{(N_i)} N_i^{-1}, \dots, a_{d_i}^{(N_i)} N_i^{-1}\}, \quad (4.31)$$

where  $d_i = |A^{N_i}|$ . For each  $i \geq 1$ , set

$$A_*^{N_i} = A^{N_i} \setminus A^{N_{i-1}}. \quad (4.32)$$

For each stage  $N_i$  we can easily construct a subset  $B_{N_i}$  of the integers by setting

$$B_{N_k} = \bigcup_{i=1}^k \{N_{i-1} + a N_i : a \in A_*^{N_i}\} \quad (4.33)$$

and let

$$B = \bigcup_{i=1}^{\infty} B_{N_i}. \quad (4.34)$$

It is not difficult to show that the set  $B$  has upper fractional density  $\beta$  and is equidistributed of order  $\beta$  in the sense of 4.23.

## 5 Preservation of arithmetic progressions

The purpose of this section is to show that arithmetic progressions are preserved when transitioning from the integers to the continuum or vice versa, as described in the previous sections. This is not dependent on the Hausdorff- or Fourier-dimensional aspects of the sets, only on the construction. We shall frequently refer to an arithmetic progression of length  $n$  as an  $n$ AP.

**Theorem 5.1.** *A set  $A \subseteq [0, 1]$  contains an arithmetic progression of length  $n \in \mathbb{N}$  if and only if the corresponding set  $B \subseteq \mathbb{N}$  defined by 4.34 contains an arithmetic progression of length  $n$ . If a set  $A \subseteq \mathbb{N}$  contains an  $n$ AP, the set  $B$  defined by 2.8 also contains an  $n$ AP.*

The theorem is phrased in this slightly cumbersome manner because the set  $B$  obtained in section 4 is not uniquely determined by the set  $A \subseteq [0, 1]$  that is used to construct it, just as the set  $B$  obtained in section 2 is not uniquely determined by the original set  $A \subseteq \mathbb{N}$ .

**Proof.** Suppose that  $a_i$ ,  $i = 1, 2, \dots, n$  form an arithmetic progression in some subset  $A$  of  $[0, 1]$ , with common difference  $\delta$ . Suppose that the set is approximated at some stage by the dyadic rationals  $j_k^{(a_1)} 2^{-k}, \dots, j_k^{(a_n)} 2^{-k}$ ; that is, for each  $i = 1, 2, \dots, n$  we have

$$a_i - \frac{j_k^{(a_i)}}{2^k} < \frac{1}{2^k}, \quad j_k^{(a_i)} \in \mathbb{N} \quad (5.35)$$

and we also require that

$$\frac{j_k^{(a_i)}}{2^k} < a_i. \quad (5.36)$$

Furthermore, it is necessary that the approximation be fine enough so that  $j_k^{(a_i)} \neq j_k^{(a_l)}$  for  $i \neq l$ .

The approximation does not need to be with dyadic intervals. This simply makes the argument somewhat easier to follow. By the nature of the approximation, if  $k$  is sufficiently large so that the intervals are smaller than the common difference, then

$$\delta - \frac{1}{2^k} < \frac{j_k^{(a_{i+1})} - j_k^{(a_i)}}{2^k} < \delta + \frac{1}{2^k} \quad (5.37)$$

for  $i = 0, 1, \dots, n-1$ . Therefore,

$$\delta 2^k - 1 < j_k^{(a_{i+1})} - j_k^{(a_i)} < \delta 2^k + 1. \quad (5.38)$$

If  $\delta 2^k$  is an integer, we are done, and so we suppose it is not. For each  $i$ , this yields two possible integer solutions for the value of  $j_k^{(a_{i+1})} - j_k^{(a_i)}$ , namely the smallest integer greater than  $\delta 2^k$ , and the largest integer smaller than  $\delta 2^k$ . Consider the case of an arithmetic progression  $a_1, a_2, a_3 \in [0, 1]$  of length 3 with common difference  $\delta$ , and suppose without loss of generality that  $j_k^{(a_2)} - j_k^{(a_1)} < j_k^{(a_3)} - j_k^{(a_2)}$ . By the nature of the approximation it is not possible for the difference to be larger than 1, so we can assume that

$$j_k^{(a_2)} - j_k^{(a_1)} + 1 = j_k^{(a_3)} - j_k^{(a_2)}. \quad (5.39)$$

We will now show that there had to have been a previous stage of the approximation for which the arithmetic progression was preserved. In order to avoid triviality, we assume that the approximation was started with a large enough integer so that the elements of the arithmetic progression were separated – that is, no two of them ever occurred in the same interval, and that this holds for at least  $n-1$  stages before the current approximation. Consider now an earlier stage of the construction, which we can without loss take to be the previous stage. We have the obvious inequalities

$$\frac{j_{N_i}^{(a_m)}}{2^{N_i}} = \frac{2^M j_{N_{i-1}}^{(a_m)}}{2^{N_i}} + \frac{k_m}{2^{N_i}}, \quad (5.40)$$



where  $M = N_i - N_{i-1}$ ,  $m \in \{1, 2, 3\}$  and  $k_m \in \mathbb{Z}$ ,  $0 \leq k_l \leq 2^M$ . By using 5.39 and 5.40, the differences in the approximations at stage  $N_{i-1}$  are given by

$$\frac{j_{N_{i-1}}^{(a_3)}}{2^{N_{i-1}}} - \frac{j_{N_{i-1}}^{(a_2)}}{2^{N_{i-1}}} = \frac{j_{N_{i-1}}^{(a_2)}}{2^{N_{i-1}}} - \frac{j_{N_{i-1}}^{(a_1)}}{2^{N_{i-1}}} + \frac{2k_2 - k_1 - k_3}{2^{N_i}} - \frac{1}{2^{N_1}}. \quad (5.41)$$

However, the difference in endpoints of the intervals for the  $N_{i-1}$ -approximation must be an integer multiple of  $2^{N_{i-1}}$ , which implies that, if  $N_i$  is large with respect to  $N_{i-1}$ , that  $2k_2 - k_1 - k_3 - 1 = 0$ . We will therefore find an arithmetic progression  $j_{N_{i-1}}^{(a_1)}, j_{N_{i-1}}^{(a_2)}, j_{N_{i-1}}^{(a_3)}$ .

We use the preservation of 3-arithmetic progressions to show that progressions of any length will be preserved. The key to this is that, by the above, a 3AP in the  $N_i$ -approximation implies the existence of one in the  $N_{i-1}$ -approximation. Now consider an AP  $a_1, a_2, a_3, \dots, a_n$  in  $[0, 1]$ , and from this consider the  $n-2$  3-progressions  $\{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, \dots, \{a_{n-2}, a_{n-1}, a_n\}$ . We can simply apply the above reasoning to show that, if  $i$  and  $N_i$  are large enough, we can possibly go back  $n-2$  stages of the construction to ensure each of the 3APs have a 3AP in the approximation; this will then imply that the approximation at stage  $N_{i-n+2}$  will contain an  $n$ -arithmetic progression. Explicitly, suppose we found, in the manner of the above, a 3AP at stage  $N_{i-1}$ , which we can denote by  $\{j_{N_{i-1}}^{(a_1)}, j_{N_{i-1}}^{(a_2)}, j_{N_{i-1}}^{(a_3)}\}$ . If  $j_{N_{i-1}}^{(a_2)}, j_{N_{i-1}}^{(a_3)}$  and  $j_{N_{i-1}}^{(a_4)}$  do not form a 3AP, we descend a level to ensure that  $\{j_{N_{i-2}}^{(a_2)}, j_{N_{i-2}}^{(a_3)}, j_{N_{i-2}}^{(a_4)}\}$  do. This descent will however preserve the first AP, that is,  $\{j_{N_{i-2}}^{(a_1)}, j_{N_{i-2}}^{(a_2)}, j_{N_{i-2}}^{(a_3)}\}$  will still form an AP, albeit with an altered common difference. Since this difference is the same as for the progression  $\{j_{N_{i-2}}^{(a_2)}, j_{N_{i-2}}^{(a_3)}, j_{N_{i-2}}^{(a_4)}\}$ , we have an AP of length 4 in  $\{j_{N_{i-2}}^{(a_1)}, j_{N_{i-2}}^{(a_2)}, j_{N_{i-2}}^{(a_3)}, j_{N_{i-2}}^{(a_4)}\}$ . By simply descending far enough, we find an AP of length  $n$ . This establishes that the set  $B$  defined by 4.34 will contain an  $n$ AP.

We now turn to the reverse. Let, for some subset  $A$  of the positive integers,  $a_i, i = 1, 2, \dots, n$  form an arithmetic progression of length  $n$ , with constant difference  $\delta$ . Suppose that these are all smaller than some integer  $N_1$  and that the construction of the continuum version of the set  $A$  is achieved with an increasing sequence of positive integers  $N_1, N_2, \dots$ . The  $N_1$  approximation of  $A$  therefore has the numbers  $a_i/N_1, i = 1, 2, \dots, n$ , in arithmetic progression; however, there is no guarantee that these numbers will be retained in the limit construction. Rather, we know that there will always be numbers determined by this arithmetic progression, no matter the size of the subdivision of  $[0, 1]$ . Per the construction, there will be numbers in the limit that can be expressed by

$$\lim_{n \rightarrow \infty} \left( \frac{a_i}{2^{N_1}} + \frac{a_i}{2^{N_2}} + \frac{a_i}{2^{N_3}} + \dots + \frac{a_i}{2^{N_n}} \right) \quad (5.42)$$

for every  $i = 1, 2, \dots, k$ . Clearly, the constant difference between the image of the terms becomes

$$\delta \lim_{n \rightarrow \infty} (2^{-N_1} + 2^{-N_2} + \dots + 2^{-N_n}) > 0. \quad (5.43)$$

Thus, the constant difference is retained, and arithmetic progressions do not collapse into single points through either side of the construction. This establishes two parts of the proof. Firstly, since a given set  $A \subseteq [0, 1]$  can be recovered from the set  $B \subseteq \mathbb{N}$ , this ensures that the sufficiency condition of the first part of the theorem is satisfied, and secondly it shows that the subset of  $[0, 1]$  obtained by the construction 2.8 will contain an  $n$ AP if the originating set does.  $\square$

This in fact establishes a close equivalence of Theorem 1.2 in [5] and the Theorem 3.1 of [7].

## 6 Probabilistic Salem sets

Salem sets crop up regularly in the study of stochastic processes, as can be seen in, for instance, [2] and [13]. In the light of the results in section 4 the reason for this, at least in the cases where the processes have independent increments, is that probabilistic choices usually yield equidistributed sequences in compact sets. The result is established in [9], but is implicit in the main theorem and not explicitly stated, as it was

formulated specifically to apply to the examination of Brownian rapid points. We state the general theorem here, independent of the theory of Brownian motion.

Although we seemingly refer to constructing a single set, it is intended that each stage be understood as part of the construction of a random set. Suppose that  $0 < \beta < 1$  is given and let  $N_1, N_2, \dots$  be an increasing sequence of positive integers. At stage 1 of the construction, we divide the interval  $[0, 1]$  into  $N_1$  equal pieces, and label an interval as “white” with probability  $p_1 = N_1^{-\beta+o(1)}$ , where  $0 < \beta \leq 1$ . The union of all white intervals is denoted by  $A_1$ . The remaining intervals are to be labelled “black”. At stage  $i + 1$ , subdivide  $[0, 1]$  into  $N_1 N_2 \cdots N_{i+1}$  equal intervals. Select, from each interval of length  $(N_1 N_2 \cdots N_i)^{-1}$  contained in  $A_i$ , subintervals of length  $(N_1 N_2 \cdots N_{i+1})^{-1}$  with probability  $(N_1 N_2 \cdots N_{i+1})^{-\beta+o(1)}$ . The union of intervals so obtained is denoted by  $A_{i+1}$ . Thus, at each stage we are considering a Bernoulli process in which the probability is determined by the interval lengths, and which is equipped with the usual binomial distribution. We now set

$$A = \bigcap_{i=1}^{\infty} A_i. \quad (6.44)$$

This type of construction is generally known as a *limsup random fractal*; see for instance [4]. We then have the following theorem:

**Theorem 6.1.** *The random set  $A$  construct by the method above will have Hausdorff dimension  $1 - \beta$ , almost surely.*

The proof of this for the rapid points of algorithmically random Brownian motion appears in [8]. The proof relies only on the magnitude of the probability and the construction as described above, rather than properties specific to Brownian motion, and is therefore applicable to any such set which derives from successive Bernoulli processes. The proof of the following is somewhat more involved, which is why we will summarise some of the key steps as found in [9]

**Theorem 6.2.** *The random set  $A$  will have Fourier dimension  $1 - \beta$ , almost surely.*

The key step in the proof is the following lemma, which is iterated to show that the sequence of measures constructed converge to something with the desired decay. In order for the convergence to a probability measure to take place it is necessary to use Theorem 6.1, which provides a good illustration of how Hausdorff dimension influences the Fourier dimension in Salem sets. We let  $\mu_0$  denote Lebesgue measure on  $[0, 1]$  and let  $\chi_1$  denote the indicator function of the first set  $A_1$  obtained in the construction of  $A$ . For any Borel  $B \subseteq [0, 1]$ , let  $\xi_1(B) = 1$  if there is some  $x \in B$  such that  $\chi_1(x) = 1$ , and 0 otherwise. We now define the measure

$$\mu_1(B) = p^{-1} \xi_1(B) \mu_0(B) \quad (6.45)$$

on Borel subsets of  $[0, 1]$ .

**Lemma 6.3.** [9] *Given  $\varepsilon_1 > 0$ , for a large enough choice of  $N_1$  in the construction of the random set  $A$ , the inequality*

$$|\widehat{\mu_1}(u) - \widehat{\mu_0}(u)| < \varepsilon_1 |u|^{\frac{1}{2}(\beta-1)} \quad (6.46)$$

*holds for all  $|u| > 1$ , with probability approaching 1 as  $N_1 \rightarrow \infty$ .*

This lemma is then iterated over the construction to obtain the same inequality for the measures  $\mu_2, \mu_3, \dots$ . By choosing the constant  $\varepsilon_i$  at the  $i$ th iteration of the lemma to be small enough compared to  $\varepsilon_{i-1}$ , we can ensure that the Fourier-Stieltjes transform of the limit measure  $\mu$  satisfies the desired decay conditions. For example, letting  $\varepsilon = 2^{-i}$  is sufficient. In light of the results in this paper, we then have the following corollary:

**Corollary 6.4.** *Suppose  $A_1 \supset A_2 \supset A_3 \supset \dots$  is a sequence of random sets, where  $A_i$  is a set whose members are intervals  $I_k^{A_i} \subset [0, 1]$  of equal length, each of the form  $I_k^{A_i} = [k/N_i, (k+1)/N_i]$  for some  $k \in \{0, 1, 2, \dots, N_i - 1\}$ , and  $N_i$  is some increasing sequence of positive integers,  $i \in \mathbb{N}$ . Suppose further that each element of  $A_i$  occurs with probability  $N_i^{-\beta+o(1)}$ . Then the initial points of the intervals in  $A_i$  will be asymptotically equidistributed of order  $1 - \beta$ , almost surely.*

Theorems 6.1 and 6.2 therefore assure us of finding a random set to be Salem, almost surely, whenever it can be described as a limsup fractal constructed in the way above through the use of independent intervals, with probabilities a power of the length of the interval, in addition to an error term which tends to 0 as the intervals become smaller. Examples of such sets are the rapid points of Brownian motion and the functionally determined rapid points of Brownian motion [9]. Corollary 6.4 indicates that this can be regarded as an equidistribution phenomenon occurring in Bernoulli processes.

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